MULTIPLICITY OF JET SCHEMES OF MONOMIAL SCHEMES

CORNELIA YUEN

Abstract. This article studies jet schemes of monomial schemes. They are known to be equidimensional but usually are not reduced [13]. We thus investigate their structure further, giving a formula for the multiplicity along every component of the jet schemes of a general reduced monomial hypersurface (that is, the case of a simple normal crossing divisor).

1. Introduction

Let X be a scheme of finite type over an algebraically closed field k of characteristic zero. An m-jet on X is a k-morphism

$$\operatorname{Spec} k[t]/(t^{m+1}) \to X.$$

We can think of such a jet as the choice of a point x in X, together with the choice of a tangent direction, a second order tangent direction, and so forth up to an m^{th} order tangent direction at x. The set of all m-jets on X forms a scheme in a natural way. This is the m^{th} jet scheme $\mathcal{J}_m(X)$.

The first serious study of arc spaces appears to have been undertaken by Nash in the 60's [18]. Nash was interested in whether the singularities of a variety X could be reflected in its arc space. Interest in jet schemes and arc spaces was recently rejuvenated by Kontsevich's introduction of motivic integration in his 1995 lecture at Orsay [14]. Since then the theory has been further developed by Batyrev [1, 2], Denef and Loeser [6, 7, 9], and others. For good surveys on motivic integration, see [3, 4, 8, 15, 19]. Inspired by these developments but in a somewhat different direction, Mustaţă and his collaborators later tied jet schemes to the study of singularities and invariants in birational geometry. For results in this direction, see [5, 10, 11, 12, 16, 17].

However, there has been little study of the explicit scheme structure of jet schemes, even in simple cases. Goward and Smith [13] began a study of the jet schemes of monomial varieties by giving an explicit description of the irreducible components of the jet schemes of monomial schemes, see Section 2. In this paper, we continue this project by finding a formula for the multiplicity of these jet schemes along each of their irreducible components.

I would like to thank Karen Smith for suggesting this problem, and for many valuable discussions. Also I am grateful to Hai Long Dao for his inspiring idea.

2. Statement of main theorem

First we recall how the set of all m-jets on an affine variety forms a scheme.

Let $X \subseteq \mathbb{A}^n$ be a closed subscheme, say $X = \operatorname{Spec} k[x_1, \ldots, x_n]/I$. An m-jet of X is equivalent to a k-algebra homomorphism

$$\phi : k[x_1, \ldots, x_n]/I \to k[t]/(t^{m+1}).$$

The map ϕ is clearly determined by the images of the generators x_i under ϕ, say $\phi(x_i) = x_i^{(0)} + x_i^{(1)}t + \ldots + x_i^{(m)}t^m$. However, there are constraints on the $x_i^{(j)}$'s coming from the relations on the x_i's. Indeed, fixing a set of generators $f_1, \ldots, f_d$ for the ideal I, we must have $\phi(f_k) = 0$ for each $k = 1, \ldots, d$, or that

$$(\dagger) \qquad f_k(x_1^{(0)} + \ldots + x_1^{(m)}t^m, \ldots, x_n^{(0)} + \ldots + x_n^{(m)}t^m) = 0 \quad \text{in } k[t]/(t^{m+1}).$$

Rearranging (†) so as to collect like terms of powers of t, we have

$$f_k^{(0)} + f_k^{(1)}t + \ldots + f_k^{(m)}t^m$$

where each $f_k^{(l)} \in k[x_1^{(0)}, \ldots, x_1^{(m)}, \ldots, x_n^{(0)}, \ldots, x_n^{(m)}]$. Then $\mathcal{J}_m(X)$ is the subscheme of $\operatorname{Spec} k[x_i^{(j)}] = \mathcal{J}_m(\mathbb{A}^n)$ defined by the polynomials $f_k^{(l)}$, as k ranges from 1 to d and l ranges from 0 to m. When $X \subseteq \mathbb{A}^n$ is an affine scheme, we let $J_m(X)$ denote the defining ideal of the jet scheme $\mathcal{J}_m(X)$ as a subscheme of $\mathcal{J}_m(\mathbb{A}^n) \cong \mathbb{A}^{n(m+1)}$.

For general properties of jet schemes, see [3, 16].

Now let $X \subseteq \mathbb{A}^n$ be a reduced monomial hypersurface defined by a single monomial $x_1 \cdots x_r$. Its jet scheme $\mathcal{J}_m(X)$ is the closed subscheme of

$$\mathcal{J}_m(\mathbb{A}^n) = \operatorname{Spec} k[x_1^{(0)}, \ldots, x_1^{(m)}, \ldots, x_n^{(0)}, \ldots, x_n^{(m)}],$$

cut out by polynomials $g_0, \ldots, g_m$ where g_k is the coefficient of t^k in the product

$$\prod_{i=1}^{r} (x_i^{(0)} + x_i^{(1)}t + \ldots + x_i^{(m)}t^m).$$

In other words, the ideal

$$J_m(X) \subseteq R = k[x_1^{(0)}, \ldots, x_1^{(m)}, \ldots, x_n^{(0)}, \ldots, x_n^{(m)}]$$

has generators of the form

$$g_k \quad = \quad \sum x_1^{(i_1)} x_2^{(i_2)} \cdots x_r^{(i_r)}$$

where $\sum i_j = k$ and $0 \leq i_j \leq m$.

Theorem (Goward-Smith)**.** *With X as above, the minimal primes of $J_m(X)$ are of the form*

$$(\star) \qquad P(m; t_1, \ldots, t_r) = (x_1^{(0)}, \ldots, x_1^{(t_1-1)}, \ldots, x_r^{(0)}, \ldots, x_r^{(t_r-1)})$$

where $0 \leq t_i \leq m+1$ and $\sum t_i = m+1$. (Here, we adopt the convention that the value $t_i = 0$ means the variable x_i does not appear at all.)

In particular, $\mathcal{J}_m(X)$ is pure dimensional — each component has codimension $m+1$. This gives a complete understanding of the irreducible components of the reduced jet schemes of X, but no more information about the scheme structure. The following theorem gives some insight into this scheme structure:

Main Theorem. *With notation as above, for* $X = \operatorname{Spec} k[x_1, \dots, x_n]/(x_1 \cdots x_r)$, *the multiplicity of* $\mathcal{J}_m(X)$ *along* $P(m; t_1, \dots, t_r)$ *is*

$$\frac{(m+1)!}{t_1! \cdots t_r!}.$$

3. Proof of main theorem

We begin with two reductions. Fix a minimal prime $P = P(m; t_1, \dots, t_r)$ of $J_m(X)$ as described in the theorem above and let $R(m; t_1, \dots, t_r) = R_P$. We want to find $\ell(R(m; t_1, \dots, t_r)/J_m(X))$.

(1) Let $R' = k[x_i^{(j)}]$, $1 \le i \le r$, $j \ge 0$, be a polynomial ring in infinitely many indeterminates. Also let $R'(m; t_1, \dots, t_r) = R'_P$. Since length is unaffected by completion, one can check then $\ell(R(m; t_1, \dots, t_r)/J_m(X)) = \ell(R'(m; t_1, \dots, t_r)/J_m(X))$. So we may replace R by R'.

(2) Suppose that one of the t_i, say t_r, is zero — that is, suppose that x_r does not appear in P at all. Notice that we can rewrite the generators of $J_m(X)$ as

$$g_k = \sum_{q=0}^{k} x_r^{(k-q)} h_q \qquad \text{with} \qquad h_q = \sum x_1^{(i_1)} \cdots x_{r-1}^{(i_{r-1})}$$

where $\sum i_j = q$ and $0 \le i_j \le m$. In other words,

$$\begin{aligned} g_0 &= x_r^{(0)} h_0 \\ g_1 &= x_r^{(1)} h_0 + x_r^{(0)} h_1 \\ g_2 &= x_r^{(2)} h_0 + x_r^{(1)} h_1 + x_r^{(0)} h_2 \\ &\text{etc.} \end{aligned}$$

Because $x_r^{(0)}$ is a unit in $R(m; t_1, \dots, t_{r-1}, 0)$, it follows that

$$J_m(X) R(m; t_1, \dots, t_{r-1}, 0) = (h_0, \dots, h_m) R(m; t_1, \dots, t_{r-1}, 0).$$

This means that

$$\begin{aligned} \ell(R(m; t_1, \dots, t_{r-1}, 0)/J_m(X)) &= \ell(R(m; t_1, \dots, t_{r-1}, 0)/(h_0, \dots, h_m)) \\ &= \ell(R(m; t_1, \dots, t_{r-1})/J_m(X')) \end{aligned}$$

where $X' = \operatorname{Spec} k[x_1, \dots, x_n]/(x_1 \cdots x_{r-1})$. So we may assume all t_i are positive.

To prove the main theorem, we also need the following lemma:

Lemma. *Let* R *be an arbitrary ring and* $x_1, \dots, x_r$ *nonunits in* R. *If* $x_1 \cdots x_r$ *is a nonzerodivisor on* R, *then* $\ell\left(R/(x_1 \cdots x_r)\right) = \sum_{i=1}^{r} \ell\left(R/(x_i)\right)$.

Proof. The proof is a simple induction on r. □

Proof of Main Theorem. We proceed by induction on m. A minimal prime of $J_0(X)$ has the form $(x_k^{(0)})$ and $\ell\left(k[x_i^{(j)}]_{(x_k^{(0)})}/(x_1^{(0)}\cdots x_r^{(0)})\right) = \ell\left(k[x_i^{(j)}]_{(x_k^{(0)})}/(x_k^{(0)})\right) = 1$. This completes the $m=0$ case. Now we assume that the assertion is true for $m-1$ and consider the multiplicity of $\mathcal{J}_m(X)$ along $P(m; t_1, \ldots, t_r)$.

Note that

$$\begin{aligned}\mathrm{ht}(J_m(X)R(m;t_1,\ldots,t_r)) &= \dim(R(m;t_1,\ldots,t_r))\\ &= \sum_{i=1}^r t_i \qquad (\text{by } \star)\\ &= m+1,\end{aligned}$$

which is the number of generators of $J_m(X)$. So the elements $g_0, \ldots, g_m$ form a regular sequence on $R(m; t_1, \ldots, t_r)$. Now let $S = R(m; t_1, \ldots, t_r)/(g_1, \ldots, g_m)$. Then $\ell(R(m; t_1, \ldots, t_r)/J_m(X)) = \ell(S/(g_0))$.

Since $g_0 = x_1^{(0)} \cdots x_r^{(0)}$ is a nonzerodivisor on S, by the Lemma, $\ell\left(S/(x_1^{(0)}\cdots x_r^{(0)})\right)$ $= \sum_{n=1}^r \ell\left(S/(x_n^{(0)})\right)$. To know what $\ell\left(S/(x_n^{(0)})\right)$ is, we need the following crucial result:

Claim. *The ring $S/(x_n^{(0)})$ is isomorphic to $R(m-1; t_1, \ldots, t_n - 1, \ldots, t_r)/J_{m-1}(X)$.*

Proof. Modulo $x_n^{(0)}$, the generators g_k $(1 \le k \le m)$ become

$$\widetilde{g_k} = \sum x_1^{(i_1)} x_2^{(i_2)} \cdots x_r^{(i_r)}$$

where $\sum i_j = k$, $0 \le i_j \le m$ if $j \ne n$ and $1 \le i_n \le m$.

By a change of variables, replacing $x_n^{(q)}$ by $x_n^{(q-1)}$ for all $q \ge 1$ and fixing the rest, $\widetilde{g_k}$ becomes g_{k-1}. This tells us

$$\begin{aligned}S/(x_n^{(0)}) &= R(m;t_1,\ldots,t_r)/(\widetilde{g_1},\ldots,\widetilde{g_m},x_n^{(0)})\\ &\cong R(m-1;t_1,\ldots,t_n-1,\ldots,t_r)/(g_0,\ldots,g_{m-1})\\ &= R(m-1;t_1,\ldots,t_n-1,\ldots,t_r)/J_{m-1}(X).\end{aligned}$$

Therefore, by the induction hypothesis, we have

$$\begin{aligned}\ell(R(m;t_1,\ldots,t_r)/J_m(X)) &= \sum_{n=1}^r \frac{m!}{t_1!\cdots(t_n-1)!\cdots t_r!}\\ &= \frac{m!(t_1+\ldots+t_r)}{t_1!\cdots t_r!}\\ &= \frac{(m+1)!}{t_1!\cdots t_r!}.\end{aligned}$$

□

References

[1] Victor V. Batyrev. Stringy Hodge numbers of varieties with Gorenstein canonical singularities. In *Integrable systems and algebraic geometry (Kobe/Kyoto, 1997)*, pages 1–32. World Sci. Publishing, River Edge, NJ, 1998.

[2] Victor V. Batyrev. Non-Archimedean integrals and stringy Euler numbers of log-terminal pairs. *J. Eur. Math. Soc. (JEMS)*, 1(1):5–33, 1999.

[3] Manuel Blickle. A short course on geometric motivic integration. *arXiv:math.AG/0507404*.

[4] Alastair Craw. An introduction to motivic integration. In *Strings and geometry*, volume 3 of *Clay Math. Proc.*, pages 203–225. Amer. Math. Soc., Providence, RI, 2004.

[5] Tommaso de Fernex, Lawrence Ein, and Mircea Mustaţă. Bounds for log canonical thresholds with applications to birational rigidity. *Math. Res. Lett.*, 10(2-3):219–236, 2003.

[6] Jan Denef and François Loeser. Germs of arcs on singular algebraic varieties and motivic integration. *Invent. Math.*, 135(1):201–232, 1999.

[7] Jan Denef and François Loeser. Motivic exponential integrals and a motivic Thom-Sebastiani theorem. *Duke Math. J.*, 99(2):285–309, 1999.

[8] Jan Denef and François Loeser. Geometry on arc spaces of algebraic varieties. In *European Congress of Mathematics, Vol. I (Barcelona, 2000)*, volume 201 of *Progr. Math.*, pages 327–348. Birkhäuser, Basel, 2001.

[9] Jan Denef and François Loeser. Motivic integration, quotient singularities and the McKay correspondence. *Compositio Math.*, 131(3):267–290, 2002.

[10] Lawrence Ein, Robert Lazarsfeld, and Mircea Mustaţă. Contact loci in arc spaces. *Compos. Math.*, 140(5):1229–1244, 2004.

[11] Lawrence Ein and Mircea Mustaţă. Inversion of adjunction for local complete intersection varieties. *Amer. J. Math.*, 126(6):1355–1365, 2004.

[12] Lawrence Ein, Mircea Mustaţă, and Takehiko Yasuda. Jet schemes, log discrepancies and inversion of adjunction. *Invent. Math.*, 153(3):519–535, 2003.

[13] Russell Goward and Karen Smith. The jet schemes of a monomial scheme. *Communications in Algebra (to appear)*, 2006.

[14] Kontsevich. Lecture at orsay, 1995.

[15] Eduard Looijenga. Motivic measures. *Séminaire Bourbaki*, 874, 2000.

[16] Mircea Mustaţă. Jet schemes of locally complete intersection canonical singularities. *Invent. Math.*, 145(3):397–424, 2001.

[17] Mircea Mustaţă. Singularities of pairs via jet schemes. *J. Amer. Math. Soc.*, 15(3):599–615 (electronic), 2002.

[18] John F. Nash, Jr. Arc structure of singularities. *Duke Math. J.*, 81(1):31–38 (1996), 1995.

[19] Willem Veys. Arc spaces, motivic integration and stringy invariants. *arXiv:math.AG/0401374*.

Department of Mathematics, University of Michigan, Ann Arbor, MI 48109, USA
E-mail address: oyuen@umich.edu